

- The general linear diff. eqn. of the n th order is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = R$$

If P_1, P_2, P_3 are constants and R the function of x only. Then the above equation is called linear diff. eqn. with constant coefficients.

- working rule for finding Complementary Function (C.F.) and Particular Integral of l. d. eqn.

- If n th order linear diff. eqn. of be

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = R \quad \dots \dots (1)$$

- Working Rule:—

(i) If R.H.S. of (1) be zero then the ^{so obtained} reduced equation is called C.F. of (1) and is the complete solution of so obtained eqn. [i.e. if $R=0$ of (1)].

(ii) Any particular solution of (1) is called Particular Integral (P.I.) of (1).

(iii) The required solution of the given equation will be addition of C.F. and P.I.

i.e. the general solution = C.F. + P.I.

For finding solution of the given equation, first of all we find C.F. when R.H.S. is zero of the given eqn.

- Consider n th order L.D. Eqn be

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n(x) = R \quad (1)$$

- The symbol D, D^2, D^3, \dots are called operators. The index of D indicates the number of times the operation of differentiation must be carried out. For example $D^3 x^4$ shows that we must differentiate x^4 three times.

Thus, $D^3 x^4 = 24x$

- Negative index of D :—

D^{-1} is an equivalent to integration. For example

$$D^{-1} x = \int x dx = \frac{x^2}{2}$$

But it is important that to note that the main object of D^{-1} is to find an integral but not the complete integral.

$$D^{-2}(x) = \int [\int x dx] dx = \int \frac{x^2}{2} dx = \frac{x^3}{2 \times 3} = \frac{x^3}{6}$$

- Now $\frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2(x)$ can be written

in symbolic form as

$$D^2 y + P_1 D y + P_2 y = (D^2 + P_1 D + P_2) y = f(D) y$$

where $f(D)$ or $(D^2 + P_1 D + P_2)$ is an operator, which operates on y .

Consider the linear differential eqn of nth order be

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = R \dots (1)$$

(1) can be written also

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = R$$

$$\Rightarrow (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = R \dots (2)$$

$$\Rightarrow f(D) y = R \dots (3)$$

Where $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n$

and $f(D)$ now acts as an operator and operates on y to yield R .

Eqs. (2) and (3) are called symbolic form of (1)

• Auxiliary Equation (a.e.): —

we write the a.e. of (3) as $f(D) = 0$ and solve it for D .

• Working rules for finding C.F. (Complementary Function)

We first rewrite the given equation in symbolic form like $f(D)y = R$. Then we write its a.e. namely $f(D) = 0$. Now, we solve for D . On solving the a.e. we shall get n roots of n th order equation. Three cases arise here.

- (i) Simple real roots - like as 2, 5, -1 and so on.
- (ii) Complex roots: - of the type $\alpha \pm i\beta$
- (iii) Roots of the type - $\alpha \pm \sqrt{\beta}$.

We discuss one by one.

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• Case I: — First suppose that the a.e. has n distinct (Pard-Pard) roots say $m_1, m_2, m_3, \dots, m_n$ then the C.F. is given by

$$C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \dots + C_n e^{m_n x}$$

where $C_1, C_2, C_3, \dots, C_n$ are arbitrary constants.

For example: —

If the roots be 2 and 3, then

$$\text{C.F.} = C_1 e^{2x} + C_2 e^{3x}$$

Again if the a.e. has real roots m occurring k times and further the remaining roots of the a.e. are distinct real numbers: —

$m_{k+1}, m_{k+2}, \dots, m_n$. Then

$$\begin{aligned} \text{C.F.} = & (C_1 + C_2 x + C_3 x^2 + \dots + C_k x^{k-1}) e^{m x} \\ & + C_{k+1} e^{m_{k+1} x} + C_{k+2} e^{m_{k+2} x} + \dots \end{aligned}$$

For example: —

If the roots be 2, 2, 2, -1, 3 then

$$\text{C.F.} = (C_1 + C_2 x + C_3 x^2) e^{3x} + C_4 e^{-x} + C_5 e^{3x}$$

• Case II: — Let $\alpha \pm i\beta$ be a pair of complex roots. Then corresponding part of C.F. may be written in one of the following forms: —

(a) $e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$

(b) $C_1 e^{\alpha x} \cos(\beta x + C_2)$

(c) $C_1 e^{\alpha x} \sin(\beta x + C_2)$

If $\alpha \pm i\beta$ occur - twice then

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$$C.F. = e^{\alpha x} \{ (C_1 + C_2 x) \cos \beta x + (C_3 + C_4 x) \sin \beta x \}.$$

Case III: — If a pair of the roots of a.e. be like as $\alpha \pm \sqrt{\beta}$ where β is positive, then corresponding part of C.F. is one of the following three forms —

(a) $e^{\alpha x} [C_1 \cosh(\alpha\sqrt{\beta}) + C_2 \sinh(\alpha\sqrt{\beta})]$

(b) $C_1 e^{\alpha x} \cosh(\alpha\sqrt{\beta}) + C_2$

(c) $C_1 e^{\alpha x} \sinh(\alpha\sqrt{\beta}) + C_2$

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